

# On the Calculation of Nonlinear Spinor Field Functionals

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Dynamics of quantum field theory can be formulated by functional equations. To develop a complete functional quantum theory one has to describe the physical information by functional operations only. Such operations have been defined in preceding papers. To apply these operations to physical problems, the corresponding functionals have to be known. Therefore in this paper calculational procedures for functionals are discussed. As high energy phenomena are of interest, the calculational procedures are given for spinor field functionals. Especially a method for the calculation of stationary and Fermion-Fermion scattering functionals is proposed.

The operator equations of quantum field theory can be replaced formally by functional equations of corresponding Schwinger functionals<sup>1-3</sup>. To give this formalism a physical and mathematical meaning one has to develop a complete functional quantum theory as has been proposed in a preceding paper<sup>4</sup>. Then the complete physical information has to be given by functional operations only. This programme has been realized already for free relativistic Fermi-fields<sup>5</sup>, for the functional *S*-matrix definition of nonrelativistic and relativistic Fermion-Fermion scattering<sup>6,7</sup> and for the quantum number definition of any field theory in functional space<sup>8</sup>. To apply these functional definitions and procedures to physical problems, the corresponding functionals have to be known, i. e., they have to be calculated. Therefore in this paper we discuss calculational procedures. In contrary to the preceding papers no exact proofs can be given. As no theory of solution for functional equations exists so far, the methods proposed here are guided by physical intuition and analogy. But it is to hope, that they stimulate the exact mathematical approach. As the functional formalism has the same mathematical structure for non-relativistic quantum mechanics and relativistic quantum field theory with different methods of regularization, the calculation methods can be applied to

any quantum theoretical problem. Therefore test calculations have been made already for the harmonic and anharmonic oscillator<sup>4,9-13</sup>. To be able to perform functional calculations for interesting high energy phenomena, in this paper the nonlinear spinor equation regularized by noncanonical Heisenberg quantization<sup>14</sup> is assumed to be the basic field equation. For this theory the calculation of stationary and of scattering functionals is discussed. The methods also allow a connection to the earlier N.T.D. calculations of Heisenberg and coworkers in configurational space.

## 1. Fundamentals

We discuss first the dynamical assumptions of the theory

a) A Hermitean spinor field operator  $\Psi_\alpha(x)$  can be defined<sup>15</sup> the dynamical behavior of which is given by the nonlinear equation

$$D_\alpha^\beta(x) \Psi_\beta(x) + V_\alpha^{\beta\gamma\delta} : \Psi_\beta(x) \Psi_\gamma(x) \Psi_\delta(x) : = 0. \quad (1.1)$$

*D* is the Hermitean Dirac operator and *V* the vertex operator, whose explicit expressions can be found in<sup>15</sup>. The dots mean normalordering. To simplify the spinor calculus we introduced in (1.1) co- and

<sup>1</sup> J. SCHWINGER, Proc. Nat. Acad. Sci. **37**, 452 [1951].

<sup>2</sup> D. LURIE, Particles and Fields, Interscience Publ., New York 1968.

<sup>3</sup> J. RZEWUSKI, Field Theory, Vol. 2, Iliffe Books, Ltd., London 1969.

<sup>4</sup> H. STUMPF, Z. Naturforsch. **24 a**, 188 [1969].

<sup>5</sup> H. STUMPF, Z. Naturforsch. **25 a**, 575 [1970].

<sup>6</sup> H. STUMPF, Z. Naturforsch. **24 a**, 1022 [1969].

<sup>7</sup> H. STUMPF, Z. Naturforsch. **25 a**, 795 [1970].

<sup>8</sup> H. STUMPF, K. SCHEERER, and H. G. MÄRTL, Z. Naturforsch. **25 a**, 1556 [1970].

<sup>9</sup> D. MAISON and H. STUMPF, Z. Naturforsch. **21 a**, 1829 [1966].

<sup>10</sup> W. SCHULER and H. STUMPF, Z. Naturforsch. **22 a**, 1842 [1967].

<sup>11</sup> W. SCHULER and H. STUMPF, Z. Naturforsch. **23 a**, 902 [1968].

<sup>12</sup> W. SCHULER, Z. Naturforsch. **23 a**, 1869 [1968].

<sup>13</sup> R. WEBER, Z. Naturforsch. **26 a** [1971], in Vorbereitung.

<sup>14</sup> W. HEISENBERG, An Introduction to the Unified Theory of Elementary Particles, Wiley & Sons, London 1967.

<sup>15</sup> H. P. DÜRR and F. WAGNER, Nuovo Cim. **46**, 223 [1966].



contravariant spinorindices in analogy to ordinary tensor calculus. This is possible only for Hermitean spinor fields. Details are given in <sup>5</sup>, App. I. Superindices are used everywhere, see <sup>5</sup>, Appendix I.

b) In analogy to free Hermitean Fermi fields the anticommutation relations for the spinor field operators are assumed to be <sup>5</sup>

$$[\Psi_\alpha(x) \Psi_\beta(x')]_{+/x_0=x'_0} = \varrho_0 \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \quad (1.2)$$

where according to noncanonical relativistic Heisenberg quantization  $\varrho_0$  disappears without making the spinor field to be a "classical" field <sup>14, 16</sup>.

c) To achieve the noncanonical behavior of the spinor field, the representation space of the field operators has to be a Hilbert space with an indefinite metric. It is assumed that the indefinite metric does not destroy the probability interpretation of the theory. This has been verified already for rigorous soluble models <sup>17, 18</sup>.

d) The quantum numbers of the theory are defined by the condition that the representation space of the field operators has to be a representation space of the inhomogeneous Lorentz-group and additional symmetry groups. We discuss only those quantum numbers resulting from the Lorentz-group. Denoting the generators of the inhomogeneous Lorentz-group by  $P_h$  and  $M_{kl}$ , their representations in the Hilbert space of nonlinear spinor theory will be written  $\mathbf{P}_h$  and  $\mathbf{M}_{kl}$ . Then the quantum numbers (mass  $m$ , linear momentum  $p$ , total spin  $s$ , spin-direction  $s_3$ ) of a state  $|a\rangle$  in this space are defined by

$$\mathbf{P}_h |a\rangle = p_h |a\rangle; \quad \mathbf{P}^2 |a\rangle = m^2 |a\rangle, \quad (1.3)$$

$$\mathbf{F}_\mu \mathbf{F}^\mu |a\rangle = s(s+1) |a\rangle; \quad \mathbf{S}_3 |a\rangle = s_3 |a\rangle \quad (1.4)$$

$$\text{with} \quad \mathbf{F}^\mu = \frac{1}{2m} \varepsilon^{\mu\nu\rho\sigma} \mathbf{P}_\nu \mathbf{M}_{\rho\sigma} \quad (1.5)$$

for  $\mathbf{S}_3$  see <sup>19</sup>.

e) According to d) any Lorentz transformation in ordinary space

$$x_\mu = a_\mu^\nu x'_\nu + b_\mu \quad (1.6)$$

causes a transformation in representation space of nonlinear spinor theory by

$$|a\rangle' = U |a\rangle \quad (1.7)$$

and by invariance arguments <sup>16, 20</sup> the Hermitean field operators have the transformation law

$$\begin{aligned} U \Psi_\alpha(x') U^{-1} &= D_\alpha^\beta \Psi_\beta(ax' + b), \\ U \Psi^\alpha(x') U^{-1} &= D_\beta^{-1\alpha} \Psi^\beta(ax' + b) \end{aligned} \quad (1.8)$$

where  $D := D(a^{-1})$  is the transformation matrix of the classical Hermitean spinor field, see <sup>5</sup>, App. I. As the spinor field is not correlated to one physical mass  $m$ , it should be noted, that (1.8) is not a trivial assumption but has to be proven <sup>8</sup>.

f) The property b) of the spinor field destroys the structure of a canonical quantum theory. Therefore the generators  $\mathbf{P}_h$  and  $\mathbf{M}_{kl}$  cannot be represented by the quantization of the corresponding classical expressions. Therefore one may assume only, that the  $\mathbf{P}_h$  and  $\mathbf{M}_{kl}$  are functionals of the spinor field operators, but their explicit expressions are not known. This difficulty is characteristic for noncanonical quantized theories. It prevents to calculate eigenvalues by current methods of canonical theory.

## 2. Generating Functionals

To avoid lengthy expressions we use the abbreviation

$$g_\alpha(z) f^\alpha(z) := \int g_\alpha(z) f^\alpha(z) dz \quad (2.1)$$

in the following.

a) Schwinger functionals are defined by

$$\begin{aligned} \mathfrak{Z}(j) &:= \langle 0 | T \exp\{i \Psi_\alpha(x) j^\alpha(x)\} | a \rangle \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \tau_n(x_1 \dots x_n) j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n) \end{aligned} \quad (2.2)$$

with

$$\tau_n(x_1 \dots x_n) := \langle 0 | T \Psi_{\alpha_1}(x_1) \dots \Psi_{\alpha_n}(x_n) | a \rangle \quad (2.3)$$

where  $T$  means timeordering and  $\langle 0 |$  is the ground-state of nonlinear spinortheory in physical Hilbert space. Due to their definition the  $\tau_n$ -functions are antisymmetric in all variables, i. e.,

$$\begin{aligned} \tau_n(x_1 \dots x_l \dots x_k \dots x_n) \\ \alpha_1 \quad \alpha_l \quad \alpha_k \quad \alpha_n \\ = -\tau_n(x_1 \dots x_k \dots x_l \dots x_n) \cdot \end{aligned} \quad (2.4)$$

<sup>16</sup> H. RAMPACHER, H. STUMPF, and F. WAGNER, Fortschr. Phys. **13**, 385 [1965].

<sup>17</sup> W. HEISENBERG, Nucl. Phys. **4**, 532 [1957].

<sup>18</sup> K. L. NAGY, State Vector Spaces with Indefinite Metric in Quantum Field Theory, P. Nordhoff, Ltd., Groningen 1966.

<sup>19</sup> H. JOOS, Fortschr. Phys. **10**, 65 [1962].

<sup>20</sup> S. S. SCHWEBER, H. A. BETHE, and F. DE HOFFMANN, Mesons and Field I, Row Peterson Comp., New York 1956.

b) In order to give the functionals a relativistic invariant resp. covariant meaning the sources  $j^a(x)$  are required to be operators in a functional space with definite transformation properties. Assuming the existence of a functional Hilbert space, which is discussed in detail in the appendix any Lorentz transformation (1.6) in ordinary space induces a similarity transformation  $V$  in functional space and the sources have to transform according to

$$\begin{aligned} V j_a(x') V^{-1} &= D_a^\beta j_\beta(a x' + b), \\ V j^a(x') V^{-1} &= D_a^{-1\alpha} j^\alpha(a x' + b). \end{aligned} \quad (2.5)$$

This causes for the field functional the transformation property

$$\mathfrak{Z}'(j) = V \mathfrak{Z}(j) V^{-1}. \quad (2.6)$$

c) To construct a complete operator algebra in functional space, we prove first that the sources  $j_a(x)$  have to be anticommuting operators, i. e.,

$$[j_a(x) j_{a'}(x')]_+ = 0. \quad (2.7)$$

This follows by means of (2.3). By a change of the variables  $(x_k, \alpha_k)$  to  $(x_{k+1}, \alpha_{k+1})$  and  $(x_{k+1}, \alpha_{k+1})$  to  $(x_k, \alpha_k)$  in (2.2) follows using (2.4)

$$\begin{aligned} \mathfrak{Z}(j) &= - \sum_n \frac{i^n}{n!} \tau_n(x_1 \dots x_n) \underset{\alpha_1}{x_1} \dots \underset{\alpha_k}{x_k} \underset{\alpha_{k+1}}{x_{k+1}} \dots \underset{\alpha_n}{x_n} \\ &\cdot j^{2_1}(x_1) \dots j^{2_{k+1}}(x_{k+1}) j^{2_k}(x_k) \dots j^{2_n}(x_n). \end{aligned} \quad (2.8)$$

For the nonlinear spinor-equation (1.1) the corresponding functional equation reads

$$[D_a^\beta(x) \partial_\beta(x) + V_\alpha^{\beta\gamma\delta} [\partial_\beta(x) \partial_\gamma(x) \partial_\delta(x) + 3 F_{\beta\gamma}(0) \partial_\delta(x)] - i \varrho_0 j_a(x)] |\mathfrak{Z}(j)\rangle = 0. \quad (2.13)$$

In noncanonical quantized theory  $F_{\beta\gamma}(0)$  and  $\varrho_0$  are assumed to disappear and so we omit these quantities in the following. Equation (2.13) can be changed into an integral equation by applying the causal Green function to (2.13). This gives

$$[\partial_x(x) + G_x^\alpha(x-x') V_\alpha^{\beta\gamma\delta} \partial_\beta(x') \partial_\gamma(x') \partial_\delta(x')] |\mathfrak{Z}(j)\rangle = 0. \quad (2.14)$$

As no special value of  $x$  is preferred Eq. (2.14) has to be averaged by a functional operator  $s^*(x)$  leading to

$$s^*(x) [\partial_x(x) + G_x^\alpha(x-x') V_\alpha^{\beta\gamma\delta} \partial_\beta(x') \partial_\gamma(x') \partial_\delta(x')] |\mathfrak{Z}(j)\rangle = 0. \quad (2.15)$$

For details of the averaging see <sup>10, 11</sup>.

e) By means of (1.8) and (1.3), (1.4) subsidiary conditions for the  $\tau_n$ -functions can be derived <sup>11</sup>. These conditions can be formulated in functional space too. Denoting the representations of the generators of the Poincare group in ordinary function space by  $P_h(x)$  and  $M_{kl}(x)$  we define the functional representation of the generators by

$$\mathfrak{P}_h := j_a(x) P_h(x) \partial^a(x), \quad \mathfrak{M}_{kl} := j_a(x) M_{kl}(x) \partial^a(x). \quad (2.16), (2.17)$$

Then if  $|a\rangle$  is a pure state with the quantum numbers defined by (1.3), (1.4) the corresponding functional states (2.2), (2.12) satisfy the subsidiary conditions

$$\mathfrak{P}_h |\mathfrak{Z}(j)\rangle = p_h |\mathfrak{Z}(j)\rangle; \quad \mathfrak{P}^2 |\mathfrak{Z}(j)\rangle = m^2 |\mathfrak{Z}(j)\rangle, \quad (2.18)$$

$$\mathfrak{G}_\mu \mathfrak{G}^\mu |\mathfrak{Z}(j)\rangle = s(s+1) |\mathfrak{Z}(j)\rangle; \quad \mathfrak{S}_3 |\mathfrak{Z}(j)\rangle = s_3 |\mathfrak{Z}(j)\rangle \quad (2.19)$$

Subtraction of (2.8) from (2.2) gives the compatibility condition (2.7) q.e.d. The set of operators  $j_a(x)$  is not complete. In order to derive functional equations we introduce differential operators

$$\partial^a(x) := \left[ \frac{\delta}{\delta j_a(x)} \right] \quad (2.9)$$

where the brackets indicate that (2.9) has nothing to do with common differential operators of commuting sources. Then the anticommutation relations are completed by

$$[j_a(x) \partial^{a'}(x')]_+ = \delta_a^{a'} \delta(x-x'), \quad (2.10)$$

$$[\partial_a(x) \partial_{a'}(x')]_+ = 0. \quad (2.11)$$

Like for spinorfields one has to distinguish carefully between co- and contravariant source quantities too. In <sup>5</sup> and in the Appendix it is shown that the conditions (2.5), (2.7), (2.10), (2.11) together with the definition of a functional groundstate  $|\varphi_0\rangle$  already determine the structure of the possible functional Hilbert spaces.

d) By means of the procedure discussed in <sup>5</sup>, App. III, functional equations can be derived for the functional statevectors

$$|\mathfrak{Z}(j)\rangle := \mathfrak{Z}(j) |\varphi_0\rangle. \quad (2.12)$$

with 
$$\mathbb{G}^u := \frac{1}{2m} \varepsilon^{\mu\nu\rho\sigma} \mathfrak{P}_\nu \mathfrak{M}_{\rho\sigma} \quad (2.20)$$

and a corresponding definition of  $\mathfrak{S}_3$  resulting from  $\mathbf{S}_3$ . For a detailed discussion of  $\mathbf{S}_3$  see <sup>8</sup>.

### 3. Normal Transforms

By omitting the  $\varrho_0$ -term in (2.13) due to noncanonical quantization information about quantization is lost. Therefore the resulting Eq. (2.14) cannot have welldefined physical solutions. It is merely a recursion formula. To restore the original content of information a statement about quantization is required, which replaces the effect of the  $\varrho_0$ -term. According to Heisenberg this is achieved by normal ordering. This reads in its functional version

$$|\mathfrak{T}(j)\rangle = e^{-jFj} |\Phi(j)\rangle \quad (3.1)$$

with

$$jFj := j^2(\xi) \langle 0 | T \Psi_\alpha(\xi) \Psi_\beta(\eta) | 0 \rangle j^\beta(\eta) \quad (3.2)$$

and the power series expansion

$$|\Phi(j)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \varphi_n(x_1 \dots x_n) j^{\alpha_1} \dots j^{\alpha_n} | \Phi_0 \rangle. \quad (3.3)$$

For free fields (3.1) becomes the functional version of the Wickrule. But for the nonlinear spinor field with noncanonical quantization it is a statement about the functional structure of the physical solutions of (2.14) to be expected. Therefore the functional (3.2) is assumed to be known a priori, as it contains the information about the noncanonical quantization. Substitution of (3.1) into (2.15) then gives the normal transformed equation

$$\tilde{s}^\kappa(x) [d_\kappa(x) + G_\kappa^\alpha(x-x') V_\alpha^{\beta\gamma\delta} \cdot d_\beta(x') d_\gamma(x') d_\delta(x')] |\Phi(j)\rangle = 0 \quad (3.4)$$

where  $\tilde{s}$  is the normal transform of  $s$  and the definition

$$d_\kappa(x) := F(xz) j^\alpha(z) + \partial_\kappa(x) \quad (3.5)$$

is used following from (I.6), (I.7). Also the subsidiary conditions can be transformed to give

$$\begin{aligned} \mathfrak{P}_h |\Phi(j)\rangle &= p_h |\Phi(j)\rangle; \\ \mathfrak{P}^2 |\Phi(j)\rangle &= s(s+1) |\Phi(j)\rangle; \end{aligned} \quad (3.6)$$

$$\begin{aligned} \mathbb{G}_\mu \mathbb{G}^\mu |\Phi(j)\rangle &= m^2 |\Phi(j)\rangle, \\ \mathfrak{S}_3 |\Phi(j)\rangle &= s_3 |\Phi(j)\rangle, \end{aligned} \quad (3.7)$$

i. e., these conditions remain invariant against the normal transformation. To solve eigenvalue problems additional informations about the eigenfunctionals are required. These are the boundary conditions of the problem. We shall show, that boundary conditions for the normal transforms can be derived. But as no exact mathematical theory of solution of (3.4) is known so far, it is not clear if these conditions are sufficient to determine the solutions uniquely. Due to symmetry properties of (1.1) resp. (2.15) or (3.4) the manifold of solutions in physical Hilbert space as well as in functional space can be divided into sectors with different baryon numbers. The lowest sector contains the one particle states of spin 1/2, i. e., dressed spin 1/2 fermions.

For the present discussion of the eigenvalue problem we consider only this sector. Of course for higher sectors similar considerations should be possible. To obtain the desired information about the Normal transforms of this sector we denote the maximal set of quantum numbers for one eigenstate in this sector by  $\mathfrak{K} := km$ , where  $m$  is the total mass of the particle and  $k := p, s$  with  $p \equiv$  total four momentum,  $s \equiv$  total spin and additional quantum numbers. We assume now that stationary solutions exist in this sector and that the corresponding mass eigenvalues are a discrete, denumerable finite set, which we denote by  $\mu_1 \dots \mu_N$ . Of course all sets of quantum numbers which can be obtained by a symmetry transformation from the set  $\{\mu\}$  are eigenvalues too. Then in physical Hilbert space there exist dressed one particle states  $|\mathfrak{p}, \sigma_k, \mu_i\rangle$  with the same quantum numbers corresponding to the eigenfunctionals of the functional version. These eigenfunctionals are defined by

$$\mathfrak{T}_{\mathfrak{K}}(j) := \langle 0 | T \exp\{i \Psi_\alpha(x) j^\alpha(x)\} | \mathfrak{K} \rangle \quad (3.8)$$

and the corresponding Normal transforms we denote by  $\Phi_{\mathfrak{K}}(j)$ . Then conclusions about this functionals can be drawn by assuming the validity of a generalized asymptotic condition for the spinor field.

g) Denoting free Hermitean spinor fields of spin 1/2 particles with mass  $\mu$  by  $\Psi_\alpha(x, \mu)$  we assume the asymptotic condition

$$\lim_{t \rightarrow \infty} \Psi_\alpha(x) = \lim_{t \rightarrow \infty} \chi_\alpha(x) \quad (3.9)$$

$$\text{with} \quad \chi_\alpha(x) := \sum_{i=1}^N \Psi_\alpha(x, \mu_i) a_i \quad (3.10)$$

to be valid in the sense of weak convergence. This condition may be modified due to effects of non-canonical quantization, but at present we are only interested in its consequences. Substitution of the field operators in the definition of the Normal transforms (3.4) and applying (4.2) we obtain

$$\lim_{\substack{t_1 = \pm\infty \\ \vdots \\ t_n = \pm\infty}} \varphi_n(x_1 \dots x_n) = \lim_{\substack{t_1 = \pm\infty \\ \vdots \\ t_n = \pm\infty}} \langle 0 | N \chi(x_1) \dots \chi(x_n) | \mathfrak{R} \rangle. \quad (3.11)$$

Observing further that it is not necessary to describe the dressed one particle state  $|\mathfrak{R}\rangle$  by the field operators  $\Psi_a(x)$ , but that it may be expressed also in the free field representation by the application of a creation operator  $a^+(\mathfrak{R}_i)$  on  $|0\rangle$  it follows from (3.11)

$$\lim_{\substack{t_1 = \pm\infty \\ \vdots \\ t_n = \pm\infty}} \varphi_n(x_1 \dots x_n) = 0, \quad n = 2, 3, \dots, \infty. \quad (3.12)$$

Therefore the asymptotic condition g) leads to simple boundary conditions (3.6) for the Normal transformed eigenfunctionals.

#### 4. Solution Procedure

To solve (3.4) in <sup>4</sup> the method of functional least squares has been proposed. As the algebra and the scalar products of weighted functionals have been derived in <sup>5</sup> one may apply a weighting factor  $\exp\{jGj\}$  to (3.4) and calculate with this factor the functional norm of (3.4). The formulas that are required are given in App. I. If  $G(x, x')$  is square integrable no divergence in the single terms occur. This method has been used successfully to prove the existence of solutions for the generalized time dependent Schrödinger equation <sup>21</sup>. As the Eq. (3.4) is tightly connected with this problem one may hope, that by this method the existence of solutions can be proven also for (3.4). But for practical calculations this method suffers by the arbitrariness of  $G(x, x')$ . Only for very high approximations one may expect, that the special choice of  $G(x, x')$  does not play an important role, whereas for low approximations occurring in practice the special  $G$  may influence the results seriously. As no experience about a suitable choice of  $G$  is present, we do not discuss this method here further, but give a more

practical approach, where it can be hoped also to prove convergence. In contrary to the method of least squares, where scattering problems and eigenvalue problems are treated equally as has been shown for test problems in quantum mechanics <sup>22</sup>, for this approach the eigenvalue and scattering calculations run differently.

##### a) Eigenvalue Calculations

To solve Eq. (3.4)  $\tilde{s}^*(x)$  has to be defined. For eigenvalue problems it is convenient to choose

$$\tilde{s}^*(x) := j^*(x) P_h(x). \quad (4.1)$$

Defining then

$$\mathfrak{D}_h(j, d) := j^*(x) P_h(x) G_\alpha^\alpha(x - x') V_\alpha^{\beta, \gamma, \delta} \cdot d_\beta(x') d_\gamma(x') d_\delta(x') \quad (4.2)$$

one obtains from (3.4)

$$j^*(x) P_h(x) \mathfrak{D}_h(j, d) | \Phi(j) \rangle = - \mathfrak{D}_h(j, d) | \Phi(j) \rangle \quad (4.3)$$

and by (2.16), (2.18)

$$[m^2 - \mathfrak{D}^h(j, d) \mathfrak{D}_h(j, d)] | \Phi(j) \rangle = 0. \quad (4.4)$$

In nonlinear spinor theory the baryon number is a good quantum number and any eigenvalue calculation of dressed particles requires to fix this number. Due to baryon number conservation then there exists a smallest  $\varphi_\varrho \neq 0$ , whereas all  $\varphi_n \equiv 0$  for  $n < \varrho$ . Therefore in this baryon sector the normal transform may be written

$$| \Phi(j) \rangle = \sum_{n=\varrho}^{\infty} \varphi_n(x_1 \dots x_n) | D_n(x_1 \dots x_n) \rangle \quad (4.5)$$

where the power functionals occurring in (4.5) are defined for  $G \equiv 0$ . Defining the projection operators

$$P_k := | D_k(z_1 \dots z_k) \rangle \delta_{\beta_1}^{\gamma_1} \delta(z_1 - y_1) \dots \delta_{\beta_k}^{\gamma_k} \delta(z_k - y_k) \langle D_k(y_1 \dots y_k) | \quad (4.6)$$

$$\Pi_k := \sum_{n=k+1}^{\infty} P_k. \quad (4.7)$$

(4.5) may be written also

$$| \Phi(j) \rangle = P_\varrho | \Phi(j) \rangle + \Pi_\varrho | \Phi(j) \rangle =: | \Phi_\varrho \rangle + | \Phi_r \rangle. \quad (4.8)$$

Substitution of (4.8) into (4.4) then gives

$$(m^2 - \mathfrak{D}^h \mathfrak{D}_h) [P_\varrho | \Phi(j) \rangle + \Pi_\varrho | \Phi(j) \rangle] = 0. \quad (4.9)$$

<sup>21</sup> O. A. LADISHENSKAJA, Mat. Sb. **39**, [81] 491 [1956].

<sup>22</sup> K. LADANYI, Nuovo Cim. **61** A, 173 [1969]; **56** A, 173 [1968].



Due to the linear independency of (4.6) and (4.7) from (4.9) follow the equations

$$\begin{aligned} \mathbf{P}_o(m^2 - \mathfrak{D}_h \mathfrak{D}^h) \mathbf{P}_o | \Phi_o \rangle \\ + \mathbf{P}_o(m^2 - \mathfrak{D}_h \mathfrak{D}^h) \mathbf{\Pi}_o | \Phi_r \rangle = 0, \\ \mathbf{\Pi}_o(m^2 - \mathfrak{D}_h \mathfrak{D}^h) \mathbf{\Pi}_o | \Phi_r \rangle \\ + \mathbf{\Pi}_o(m^2 - \mathfrak{D}_h \mathfrak{D}^h) \mathbf{P}_o | \Phi_o \rangle = 0 \end{aligned} \quad (4.10)$$

where  $\mathbf{P}_n^\alpha \equiv \mathbf{P}_a$  has been used. Eliminating  $| \Phi_r \rangle$  we obtain

$$| \Phi_r \rangle = - (m^2 - \mathbf{\Pi}_o \mathfrak{D}_h \mathfrak{D}^h \mathbf{\Pi}_o)^{-1} \cdot \mathbf{\Pi}_o(m^2 - \mathfrak{D}_h \mathfrak{D}^h) \mathbf{P}_o | \Phi_o \rangle \quad (4.11)$$

and by expansion

$$| \Phi_r \rangle = - \frac{1}{m^2} \sum_{l=0}^{\infty} \left( \frac{1}{m^2} \mathbf{\Sigma}_o \mathfrak{D}_h \mathfrak{D}^h \mathbf{\Pi}_o \right)^l \cdot \mathbf{\Pi}_o \mathfrak{D}_h \mathfrak{D}^h \mathbf{P}_o | \Phi_o \rangle \quad (4.12)$$

if one observes  $m^2 \mathbf{\Pi}_o \mathbf{P}_o = 0$ . Substitution of (4.12) into (4.10) then gives the equation

$$\begin{aligned} (m^2 - \mathbf{P}_o \mathfrak{D}_h \mathfrak{D}^h \mathbf{P}_o) | \Phi_o \rangle \\ + \frac{1}{m^2} \sum_{l=0}^{\infty} \mathbf{P}_o \mathfrak{D}_h \mathfrak{D}^h \mathbf{\Pi}_o \\ \cdot \left( \frac{1}{m^2} \mathbf{\Pi}_o \mathfrak{D}_h \mathfrak{D}^h \mathbf{\Pi}_o \right)^l \mathbf{\Pi}_o \mathfrak{D}_h \mathfrak{D}^h \mathbf{P}_o | \Phi_o \rangle = 0 \end{aligned} \quad (4.13)$$

which has to be solved. Additionally the conditions (3.6), (3.7) have to be satisfied. It has been shown in <sup>10</sup> that these conditions are just satisfied if the maximal set of quantum number operators com-

mutes with the operator of the dynamical equation and if  $| \Phi_o \rangle$  alone satisfies these conditions. As the commutativity of (3.6), (3.7) with (4.4) is secured by construction, it is sufficient to solve (4.13) alone by observing the subsidiary conditions for  $| \Phi_o \rangle$ . Therefore the entire problem of solving (4.4) is reduced to the solution of (4.13). As the kernel of (4.13) contains an infinite series in practical calculations only a finite number of terms can be considered. Observing (4.12) this procedure corresponds to the neglect of higher  $\varphi$ -functions, which may be justified by the boundary conditions (3.12) and thus leads to the usual N.T.D assumption. Of course the convergence of the Neumann series (4.12) has to be proven later. First attempts in this direction have been made already for a simple equation <sup>23</sup>.

### b) Scattering Calculations

The definition and construction of the  $S$ -matrix is tightly connected with the calculational procedure of scattering functionals. As in nonlinear spinor theory the  $S$ -matrix has been constructed so far only for Fermion-Fermion scattering <sup>7</sup> we confine the discussion of the calculational procedure to this case. To have definite conditions we consider elastic nucleon-nucleon scattering. For a one nucleon state, it is  $\varphi_1 \neq 0$  and the one nucleon equation reads therefore according to (4.13)

$$\left[ (m_N^2 - \mathbf{P}_1 \mathfrak{D}_h \mathfrak{D}^h \mathbf{P}_1) + \frac{1}{m_N^2} \sum_{l=0}^{\infty} \mathbf{P}_1 \mathfrak{D}_h \mathfrak{D}^h \mathbf{\Pi}_1 \left( \frac{1}{m_N^2} \mathbf{\Pi}_1 \mathfrak{D}_h \mathfrak{D}^h \mathbf{\Pi}_1 \right)^l \mathbf{\Pi}_1 \mathfrak{D}_h \mathfrak{D}^h \mathbf{P}_1 \right] | \Phi_1 \rangle = 0. \quad (4.14)$$

This equation can be written in configuration space

$$[m_N^2 \delta(x-y) + K(x, y)] \varphi_1(y) = 0 \quad (4.15)$$

with

$$K(x, y) := \langle D_1(x) | \left[ -\mathbf{P}_1 \mathfrak{D}_h \mathfrak{D}^h \mathbf{P}_1 + \frac{1}{m_N^2} \sum_{l=0}^{\infty} \mathbf{P}_1 \mathfrak{D}_h \mathfrak{D}^h \mathbf{\Pi}_1 \left( \frac{1}{m_N^2} \mathbf{\Pi}_1 \mathfrak{D}_h \mathfrak{D}^h \mathbf{\Pi}_1 \right)^l \mathbf{\Pi}_1 \mathfrak{D}_h \mathfrak{D}^h \mathbf{P}_1 \right] | D_1(y) \rangle. \quad (4.16)$$

From symmetry conditions follows <sup>8</sup> that  $\varphi_1(y)$  has to be a solution of the ordinary Dirac equation, i. e., it has to be

$$(i^\mu G_\alpha^\beta \partial_\mu + m_N \delta_\alpha^\beta) \varphi_{1\beta}(y) = 0. \quad (4.17)$$

As (4.15) and (4.17) has to be valid for all solutions of the functional equation it follows, that the kernel  $K(x, y)$  can be written

$$K(x, y) = {}^\mu G_\alpha^\beta k_\mu(x, z) {}^e G_\beta^\gamma k_\gamma(z, y) \quad (4.18)$$

$$\text{with } {}^\mu G_\alpha^\beta k_\mu(x, z) \equiv i^\mu G_\alpha^\beta \partial_\mu \delta(x-z). \quad (4.19)$$

<sup>23</sup> W. SCHULER and H. STUMPF, preprint.

Now we consider the two nucleon problem. For this problem we have  $\varphi_2 \neq 0$  and  $\varphi_1 \equiv 0$ . Therefore its equation reads according to (4.13)

$$\left[ (m^2 - \mathbf{P}_2 \mathfrak{D}_h \mathfrak{D}^h \mathbf{P}_2) + \frac{1}{m^2} \sum_{l=0}^{\infty} \mathbf{P}_2 \mathfrak{D}_h \mathfrak{D}^h \mathbf{\Pi}_2 \left( \frac{1}{m^2} \mathbf{\Pi}_2 \mathfrak{D}_h \mathfrak{D}^h \right)^l \mathbf{\Pi}_2 \mathfrak{D}_h \mathfrak{D}^h \mathbf{P}_2 \right] |\Phi_2\rangle = 0. \quad (4.20)$$

For the total process  $m^2$  is a conserved quantum number. As for  $t \rightarrow -\infty$  the two nucleons are assumed to be at infinite distance, the total mass of the two nucleon system is  $m = 2m_N$  and due to mass conservation we have for the entire process  $m^2 = 4m_N$ . For brevity we may write (4.20) in configuration space

$$[4m_N^2 \delta(x_1 - y_1) \delta(x_2 - y_2) + K_2(x_1 x_2 y_1 y_2)] \varphi_2(y_1 y_2) = 0 \quad (4.21)$$

with

$$K_2 := \langle D_2(x_1 x_2) | \left[ -\mathbf{P}_2 \mathfrak{D}_h \mathfrak{D}^h \mathbf{P}_2 + \frac{1}{m^2} \sum_{l=0}^{\infty} \mathbf{P}_2 \mathfrak{D}_h \mathfrak{D}^h \mathbf{\Pi}_2 \left( \frac{1}{m^2} \mathbf{\Pi}_2 \mathfrak{D}_h \mathfrak{D}^h \mathbf{\Pi}_2 \right)^l \mathbf{\Pi}_2 \mathfrak{D}_h \mathfrak{D}^h \mathbf{P}_2 \right] | D_2(y_1 y_2) \rangle. \quad (4.22)$$

Applying the equivalence (4.19) Eq. (4.21) may be written

$$\begin{aligned} & \left[ 4 \left( m_N + i^\mu G \frac{\partial}{\partial x_{1\mu}} \right) \left( m_N + i^\nu G \frac{\partial}{\partial x_{2\nu}} \right) \delta(x_1 - y_1) \delta(x_2 - y_2) \right] \varphi_2(y_1 y_2) \\ & = [-K_2(x_1 x_2 y_1 y_2) + {}^\mu G k_\mu(x_1 y_1) \delta(x_2 - y_2) + {}^\nu G k_\nu(x_2 y_2) \delta(x_1 - y_1) + k_\mu(x_1 y_1) k^\mu(x_2 y_2)] \varphi_2(y_1 y_2) \end{aligned} \quad (4.23)$$

and inversion by the free nucleon Green functions gives finally

$$\begin{aligned} \varphi_2(z_1 z_2) = & e^{i p_1 z_1 + p_2 z_2} + G_N(z_1 x_1) G_N(z_2 x_2) [-K_2(x_1 x_2 y_1 y_2) + {}^\mu G k_\mu(x_1 y_1) \delta(x_2 - y_2) \\ & + {}^\nu G k_\nu(x_2 y_2) \delta(x_1 - y_1) + k_\mu(x_1 y_1) k^\mu(x_2 y_2)] \varphi_2 \end{aligned} \quad (4.23)$$

where  $p_1$  and  $p_2$  are the four momentum vectors of the ingoing or outgoing nucleons to be scattered. Therefore the total problem is reduced to a relativistic two particle equation which has to be solved by standard methods.

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### Appendix I

In this appendix we give a short review about anticommuting source operators, as far as their properties are required for the present paper. No proofs are given. For a detailed treatment we refer to <sup>5</sup>.

According to <sup>5</sup> we introduce the Hermitean source operators  $j_a(x)$  and their functional derivatives  $\partial_a(x)$ . For these quantities we assume the anticommutation rules

$$[j_a(x) j_\beta(x')]_+ = 0, [j_a(x) \partial^\beta(x')]_+ = \delta_a^\beta \delta(x - x'), [\partial^a(x) \partial^\beta(x')]_+ = 0 \quad (I.1)$$

and the transformation properties

$$\begin{aligned} V j_a(x) V^{-1} &= D_a^\beta j_\beta(ax + b); V \partial_a(x) V^{-1} = D_a^\beta \partial_\beta(ax + b), \\ V j^a(x) V^{-1} &= D_\beta^{-1a} j^\beta(ax + b); V \partial^a(x) V^{-1} = D_\beta^{-1a} \partial^\beta(ax + b) \end{aligned} \quad (I.2)$$

$$\text{with} \quad \partial^a(x) = g^{a\beta} \partial_\beta(x); \quad j^a(x) = g^{a\beta} j_\beta(x). \quad (I.3)$$

The formulae (I.1), (I.2), (I.3) are conditions imposed on the source operator algebra. Their consistency can be shown by the construction of any representation. This has been done in <sup>5</sup>. From the construction follows

$$j_a^+(x) = \partial_a(x); \quad \partial_a^+(x) = j_a(x) \quad (I.4)$$

and the existence of a groundstate  $|\varphi_0\rangle$  with

$$\partial_a(x) |\varphi_0\rangle = 0 \quad (I.5)$$

i. e.,  $\partial_a(x)$  is a functional destruction and  $j_a(x)$  a functional creation operator. Therefore by repeated application of  $j_a(x)$  on  $|\varphi_0\rangle$  a functional Hilbert space can be constructed. But this space is not the only representation space. Introducing the weighting factor  $e^{-jGj}$  a set of possible groundstates  $e^{-jGj} |\varphi_0\rangle$  may

be generated by free variation of  $G$ . For any  $G$  the state  $e^{-iGj}|\varphi_0\rangle$  is the initial vector of a complete functional Hilbert space. To show this we observe

$$A^*(x) e^{-iGj} = e^{-iGj} \hat{\mathcal{D}}^*(x) \quad (\text{I. 6})$$

with

$$A^*(x) := 2 G^{\alpha\alpha}(x x') j_\alpha(x') + \hat{\mathcal{D}}^*(x). \quad (\text{I. 7})$$

Then it follows from (I. 5)

$$A^*(x) e^{-iGj}|\varphi_0\rangle = A^*(x)|0\rangle_G = 0. \quad (\text{I. 8})$$

Therefore  $A^*(x)$  is a generalized functional destruction operator. Its Hermitean conjugate is by means of (I. 4)

$$A_\alpha^+(x) = 2 G_{\alpha\alpha}^\times(x x') \hat{\mathcal{D}}^\alpha(x') + j_\alpha(x) \quad (\text{I. 9})$$

and from (I. 1) the anticommutation rules

$$[A^*(x) A^{\alpha'}(x')]_+ = 0, [A_\alpha^+(x) A_{\alpha'}^+(x')]_+ = 0, \\ [A^*(x) A_{\alpha'}^+(x')]_+ = 4 G_\alpha^\times(x y) G_{\alpha'}^\alpha(y x') + \delta_{\alpha\alpha'}^\times \delta(x - x') = : a_{\alpha\alpha'}^\times(x x'), \quad (\text{I. 10})$$

follow together with the transformation properties

$$V A^*(x) V^{-1} = D_\sigma^{-1\alpha} A^\alpha(a x + b); \quad V A^{\alpha+}(x) V^{-1} = D_\sigma^{-1\alpha} A^{\alpha+}(a x + b), \\ V A_\alpha(x) V^{-1} = D_\sigma^\alpha A_\alpha(a x + b); \quad V A_\alpha^+(x) V^{-1} = D_\sigma^\alpha A_\alpha^+(a x + b), \quad (\text{I. 11})$$

and

$$A^*(x) = g^{\alpha\alpha} A_\alpha(x); \quad A^{\alpha+}(x) = g^{\alpha\alpha} A_\alpha^+(x). \quad (\text{I. 12})$$

For the explicit construction of the functional Hilbert spaces we define the Hermitean base functionals

$$|I_n(x_1 \dots x_n)\rangle_G := \frac{1}{\sqrt{n!}} A^+(x_1) \dots A^+(x_n) |0\rangle_G \quad (\text{I. 13})$$

with the abbreviation  $f_{a_1}(x_1) = :f(x_1)$  and  $f^{a_1}(x_1) = :f(x_1)$ . The Hermitean conjugate reads

$${}_G\langle I_n(x_1 \dots x_n)| = \frac{1}{\sqrt{n!}} {}_G\langle 0| A(x_1) \dots A(x_n) \quad (\text{I. 14})$$

and for the scalar product of Hermitean functionals we get by (I. 8), (I. 10)

$${}_G\langle I_n(x'_1 \dots x'_n)| I_m(x_1 \dots x_m)\rangle_G = \frac{1}{n!} \delta_{mn} \sum_{\lambda_1 \dots \lambda_n} (-1)^p a(x'_1 x_{\lambda_1}) \dots a(x'_n x_{\lambda_n}) \quad (\text{I. 15})$$

which is renormalized by means of division by the unknown quantity  ${}_G\langle 0|0\rangle_G$ . In the special case of  $G \equiv 0$  we obtain from (I. 13) the Hermitean base functionals for  $j_\alpha(x)$  and  $|\varphi_0\rangle$ . For a weighted ground-state  $|0\rangle_G$  these functional states can be constructed also and lead to the power functionals

$$|D_n(x_1 \dots x_n)\rangle_G := \frac{1}{n!} j(x_1) \dots j(x_n) |0\rangle_G \quad (\text{I. 16})$$

i. e.,

$$|D_n(x_1 \dots x_n)\rangle_{G=0} \equiv |I_n(x_1 \dots x_n)\rangle_{G=0} \sqrt{n!} \quad (\text{I. 17})$$

For  $G \neq 0$  the connection between power functionals and Hermitean functionals follow by direct calculation given in <sup>5</sup>. We obtain

$$|D_n(z^1 \dots z^n)\rangle_G = P(n) \sum_{q=1}^{[n/2]} \frac{1}{2^{(n-q)/2}} \frac{1}{[(n-q)/2]!} \frac{1}{n!} \frac{(-1)^{P+n}}{\sqrt{q!}} \Delta(z^1 z'^1) \dots \Delta(z^n z'^n) \\ \cdot |I_q(z'^{\lambda_1} \dots z'^{\lambda_q})\rangle_G C(z'^{\lambda_{q+1}} z'^{\lambda_{q+2}}) \dots C(z'^{\lambda_{n-1}} z'^{\lambda_n}) \quad (\text{I. 18})$$

with

$$\Delta(x x') := [4 G(x y) G^\times(y x) - \delta(x - x')]^{-1} C(x x') := -2 G^\times(x y) a(y x') \quad (\text{I. 19})$$

For details see <sup>5</sup>. The scalar product for power functionals is given by

$$f_m^\times(z'_1 \dots z'_m) {}_G\langle \overline{D_m(z'^1 \dots z'^m)} | D_n(z^1 \dots z^n)\rangle_G f_n(z_1 \dots z_n) \\ = \sum_{q=1}^{(n-2q/2)} 2^{2q-(n+m)/2} \frac{1}{[(n-2q)/2]! [(m-2q)/2]!} \frac{(-1)^{(n+m)/2}}{(2q)!^2} \\ \times f_m^\times(z'_1 \dots z'_m) \overline{\Delta(z'^1 \xi'^1)} \dots \overline{\Delta(z'^m \xi'^m)} \Delta(z^1 \xi^1) \dots \Delta(z^n \xi^n) f_n(z_1 \dots z_n) \\ \times A(\xi_1 \xi'_1) \dots A(\xi_{2q} \xi'_{2q}) \overline{C(\xi'_{2q+1} \xi'_{2q+2})} \dots C(\xi'_{m-1} \xi'_m) \overline{C(\xi_{2q+1} \xi_{2q+2})} \dots C(\xi_{n-1} \xi_n) \quad (\text{I. 20})$$

where  $f_m, f_n$  are antisymmetric test functions.